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A hierarchy of some stochastic volatility models and correspondent Black-Scholes-Merton-Garman-like PDEs and S PDEs[1]

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Abstract: *In this paper we will build a hierarchy of some stochastic models used in derivatives modeling and we offer for each two partial differential equations (PDEs) whose solution is a pricing function of selected derivative if put some boundary conditions, like in Black–Scholes equation: one is Black-Scholes-Merton-like PDE and other is S PDE.*

Keywords: stochastic models, Black-Scholes-Merton-Garman-like PDE, S-conjecture, S PDE, hierarchy.

I. Stochastic models and stochastic volatility models

Stochastic models appears in stock modeling after revelation of similar viewing of some parts from thermodynamics and stock-dynamics: molecules interactions are similar stock-players' interactions with orders put in market. One of first model recovered from thermodynamics after Bachelier (see [Bachelier1900]) researches was arithmetic Brownian motion

$$dS(t) = \mu\, dt+\sigma\, dW(t) \tag{1.1}$$

and geometric Brownian motion (known also as Black-Scholes, or Black-Scholes-Merton model, see [Black1973]):

$$dS(t) = \mu\, S(t)\, dt+\sigma\, S(t)\, dW(t), \tag{1.2}$$

where $W(t)_{t\geq 0}$ is a standard Wiener process, μ is drift and σ is volatility. In reality, σ is not a static parameter, only market-depend: new models views volatility as a stochastic factor with similar stochastic differential equation (SDE) like support:

$$dS(t) = A(t,S(t),V(t))\, dt+B(t,S(t),V(t))\, dW_1(t) \tag{1.3}$$
$$dv(t) = C(t,S(t),V(t))\, dt+D(t,S(t),V(t))\, dW_2(t), \tag{1.4}$$

where S is support, v is volatility, t is timestamp, $W_1(t)_{t\geq 0}$ and $W_2(t)_{t\geq 0}$ are two standard Wiener ρ-correlated processes:

$$(dW_1(t))\, (dW_2(t)) = \rho\, dt, \tag{1.5}$$

and A, B, C, D are 4 algebraic expressions, model-depend.

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II. Black-Scholes-Merton-Garman-like PDE

For a generalized stochastic volatility model like:

$$dS(t) = A(t,S(t),V(t))\, dt+B(t,S(t),V(t))\, dW_1(t) \qquad (2.1)$$
$$dv(t) = C(t,S(t),V(t))\, dt+D(t,S(t),V(t))\, dW_2(t), \qquad (2.2)$$

where $W_1(t)_{t\geq 0}$ and $W_2(t)_{t\geq 0}$ are two standard Wiener ρ-correlated processes:

$$(dW_1(t))\, (dW_2(t)) = \rho\, dt \qquad (2.3)$$

we will assume as the price of one derivatives unit is f(t,S,v) and we can build a risk-free portofollio like in Black-Scholes model. In [Socaciu2009] we offer a proof that:

$$-r f / c + [(b / c) (r S / B) + (a / c) - (b / c) (A / B)] \qquad (2.4)$$

is invariant, where:

$$a = f_t + A f_S + C f_v + ½ B^2 f_{SS} + ½ D^2 f_{vv} + \rho B D f_{Sv} \qquad (2.5)$$
$$b = B f_S \qquad (2.6)$$
$$c = D f_v. \qquad (2.7)$$

If denote as:

$$\beta(S, v, t) = -r f / c + [(b / c) (r S / B) + (a / c) - (b / c) (A / B)] \qquad (2.8)$$

we can rewrite as a PDE:

$$f_t + r S f_S + (C - D \beta) f_v + ½ B^2 f_{SS} + ½D^2 f_{vv} +\rho B D f_{Sv} - r f = 0 \qquad (2.9)$$

that is a *Black-Scoles-like equation* named (for Heston model and other similar models) *Merton-Garman equation* or *Black-Scholes-Merton-Garman equation*. We will name as *Black-Scholes-Merton-Garman-like equation* in general form.

For classic Black-Scholes model:

$$A(S, v, t) = \mu \qquad (2.10)$$
$$B(S, v, t) = \sigma \qquad (2.11)$$
$$C(S, v, t) = 0 \qquad (2.12)$$
$$D(S, v, t) = 0 \qquad (2.13)$$

Black-Scholes-Merton-Garman-like equation collapses as Black-Scholes equation:

$$f_t + r S f_S + ½ \sigma^2 f_{SS} - r f = 0. \qquad (2.14)$$

Note that β is *the market price of volatility risk* (see [Altăr2003], p. 6). For Heston model, an accepted form for β(S, v, t) is:

$$\lambda(S,v,t) = \xi \sqrt{v}\, \beta(S,v,t) \qquad (2.15)$$

where λ is *price of volatility risk.* Breeden propose (see [Breeden1979], conf. with [Heston1993]):

$$\lambda(S,v,t) = \lambda\, v. \qquad (2.16)$$

Lamoureux and Lastrapes show (see [Lamoreaux1993], conf. with [Heston1993]) that for options λ is not zero, but some numerical implementation for pricing of European calls use (see function *CFCallHeston()* from codefile *cf_call_heston.C* in [Premia] software package) λ=0 in pricing with no comments.

III. A conjecture and alternate PDE

For a generalized stochastic volatility model like:

$$dS(t) = A(t,S(t),V(t))\, dt+B(t,S(t),V(t))\, dW_1(t) \qquad (3.1)$$
$$dv(t) = C(t,S(t),V(t))\, dt+D(t,S(t),V(t))\, dW_2(t), \qquad (3.2)$$

where $W_1(t)_{t\geq 0}$ and $W_2(t)_{t\geq 0}$ are two standard Wiener ρ-correlated processes:

$$(dW_1(t))\, (dW_2(t)) = \rho\, dt \qquad (3.3)$$

we will assume as the price of one derivatives unit is f(t,S,v) and we can build a risk-free portofollio like in generalized Black-Scholes model. In [Socaciu2013] we offer a proof that:

$$f_t + r\, S\, f_S + r\, v\, f_v + \tfrac{1}{2} B^2 f_{SS} + \tfrac{1}{2} D^2 f_{vv} +\rho\, B\, D\, f_{Sv} - r\, f = 0 \qquad (3.4)$$

that imply the conjecture:

$$\beta(S,v,t) = (C(S,v,t) - r\, v) / D(S,v,t) \qquad (3.5)$$

or, in λ world for Heston model:

$$\lambda(S,v,t) = (k\, (\theta - v) - r\, v) \qquad (3.6)$$

that means:

$$\lambda \approx k\theta / v - k - r. \qquad (3.7)$$

IV. Border conditions for PDEs

Like in Black-Scholes model we can build similar conditions on borders at 0 and ∞ for support and for volatility and t_{max} (contract expiration) for timestamp. Border conditions for a model like

$$dS(t) = A(t,S(t),V(t))\, dt+B(t,S(t),V(t))\, dW_1(t) \qquad (4.1)$$
$$dv(t) = C(t,S(t),V(t))\, dt+D(t,S(t),V(t))\, dW_2(t), \qquad (4.2)$$

are similar like in Heston model (see [Heston1993]) for those two PDE by unknown function f:

$f(S,v,t_{max}) = payoff(S,v)$	– contract expiration	(4.3)
$f(0,v,t) = 0$	– default of support	(4.4)
$f_S(\infty,v,t) = 1$	– as ∞ of support	(4.5)
PDE with $\{v=0\}$	– as 0 volatility	(4.6)
$f(S,\infty,t) = S$	– as ∞ of volatility	(4.7)

V. Some stochastic volatility models organized in a hierarchy

For a generalized stochastic volatility model like:

$$dS(t) = A(t,S(t),V(t))\, dt + B(t,S(t),V(t))\, dW_1(t) \tag{3.1}$$
$$dv(t) = C(t,S(t),V(t))\, dt + D(t,S(t),V(t))\, dW_2(t), \tag{3.2}$$

where $W_1(t)_{t\geq 0}$ and $W_2(t)_{t\geq 0}$ are two standard Wiener ρ-correlated processes:

$$(dW_1(t))\,(dW_2(t)) = \rho\, dt \tag{3.3}$$

we found many models with this pattern. Some particular cases are:

Id	A	B	C	D	Author(s), year, refferences
1	μS	$e^v S$	$k(\theta-v)$	ε	Scott, 1987, see [Sepp2010]
2	μS	vS	$(\alpha-\beta v)v$	γv	Scott, 1987, see [Sepp2011] and Wiggins, 1987, see [Sepp2011]
3	μS	vS	$k(\theta-v)$	εv	Hull, White, 1988, see [Sepp2010]
4	μS	vS	αv	γv	Hull, White, 1988, see [Sepp2011]
5	μS	vS	$k(\theta-v)$	ε	Stein, Stein, 1991, see [Sepp2010]
6	μS	vS	$(\alpha-\beta v)$	γ	Stein, Stein, 1991, see [Sepp2011]
7	μS	vS	$\alpha/v-\beta v$	γ	Heston, 1993, see [Sepp2011]
8	μS	vS	$\alpha v-\beta v^2$	γv^2	Lewis, 2000, see [Sepp2011]
9	μS	$\sigma_L(S,t)v^{½}S$	$k(\theta-v)$	$\xi v^{½}$	Lipton, 2001, see [Sepp2010]
10	0	vS^{β}	0	αv	Hagan, 2002, see [Hagan2002]

For Lipton class, we found next particular cases:

Id	Local volatilitility (σ_L)	Author(s), year, refferences
11	$\sigma_L(S,t)$	Lipton, 2001, see [Sepp2010]
12	$1+\alpha(S-S_0)+\beta(S-S_0)^2$	see [Kluge2002], p. 4.
13	1	Heston, 1993, see [Heston1993]

VI. Conclusions and further works

We have two different similar PDEs with same border conditions for all stochastic models with stochastic volatility. Can we shown that one of derivation is bad? Both construction appear to be correct. Or must broke λ philosophy? Closed form for some models (like Heston) are based on Black-Scholes-Merton-Garman-like equation and can not be used as reference, only with caution (see null λ in one of famous numeric implementation in [Premia]). Monte Carlo simulation can be used only carefully, because we have confidence interval and discretization methods' convergence. As further work we plan to obtain a closed form solution for S PDE for Heston model and compare with numerical Monte Carlo simulations with different time discretizations.

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